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ON THE SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR FIRST-ORDER NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS ON TIME SCALES

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Abstract

In this work we present some new results concerning the existence of solutions for first-order nonlinear integro-differential equations on time scales with boundary value conditions. Our methods to prove the existence of solutions involve new differential inequalities and classical fixed-point theorems.

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1. Introduction and Preliminaries

In order to unify results from the differential calculus with results from the difference calculus, in 1990 Hilger [13] created the time scale calculus by generalizing the definition of derivative and integral to time scales. Now the study of dynamic equations on time scales has become an area of mathematics receiving a lot of attention, see monographs [2, 3, 16]. As is known, integro-differential equations find many applications in various mathematical problems, see Corduneanu [5], Guo et al. [10] and references therein for details. For the recent developments involving existence of solutions to BVPs for integro-differential equations, impulsive integro-differential equations and integral-differential equations on time scales we can refer to [1, 7, 9, 11-14, 17-18, 21-24, 27-29]. So far the main method appeared in the references to guarantee the existence of solutions is the method of upper and low solutions. Motivated by the ideas in the recent works [25-26], we come up with a new approach to ensure the existence of at least one solution for certain family of first-order nonlinear integro-differential equations with periodic boundary value conditions or antiperiodic boundary value conditions. Our methods involve new differential inequalities and the classical fixed-point theory. The traditional method, method of upper and low solutions, although has been proven effective to tackle scalar-valued integro-differential equations, it is rather cumbersome to apply to large systems of equations. This paper at least presents an option to deal with the BVPs for certain family of large systems.

Let $\mathbb{T}$ be a time scale (any nonempty closed subset of the real numbers $\mathbb{R}$ with order and topological structure defined in a canonical way). Then we introduce some definitions and lemmas which can be found in [2, 3, 13, 16].

Definition 1.1. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined as $\sigma(t) = \inf\{s \in \mathbb{T};\ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T}; s < t\}$ are called *jump operators*. The mapping $\mu(t) : \mathbb{T} \to R^{+}$ defined by $\mu(t) = \sigma(t) - t$ is called *graininess*.

Definition 1.2. A mapping $f : \mathbb{T} \to X$ is said to be differentiable at $t \in \mathbb{T}$, if there exists an $\alpha \in X$ such that for any $\varepsilon > 0$ there exists a neighborhood N of t satisfying $|f(\sigma(t)) - f(s) - (\sigma(t) - s)\alpha| \leq \varepsilon|\sigma(t) - s|$ for all $s \in N$.

Lemma 1.1 [2, p. 8]. *Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are differentiable at* $t \in \mathbb{T}^k$. *Then the product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at t with*

$$(f + g)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

where

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty; \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1.3. A mapping $f : \mathbb{T} \to X$, where X is a Banach space, is called rd-continuous if

(i) it is continuous at each right-dense $t \in \mathbb{T}$;

(ii) at each left-dense point the left-side limit $f(t^-)$ exists.

Definition 1.4. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0 \text{ for } t \in \mathbb{T}^k.$$

The set of all regressive and rd-continuous functions will be denoted in this paper by $\mathcal{R}(\mathbb{T}, \mathbb{R})$. For two functions $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ define a plus $\oplus$ and a minus $\ominus$ by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t),$$

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)}.$$

For $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, Hilger [13] proved $e_p(t, t_0) = \exp(\int_{t_o}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau)$ is the unique solution of

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1,$$

where $\xi_h(z) = \frac{1}{h}\operatorname{Log}(1 + zh)$.

Remark 1.1. If $p(t) \geq 0$ for $t \geq t_0$, clearly $1 + \mu(t)p(t) \geq 1$. Therefore $\xi_{\mu(\tau)}(p(\tau)) \geq 0$ and so $e_p(t, t_0) \geq 1$.

Lemma 1.2 [2, p. 62]. *If* $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, *then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

(iii) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;

(iv) $e_p(t, r)e_p(r, s) = e_p(t, s)$;

(v) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$.

Lemma 1.3 [28]. *Assume that* $\{f_n\}_{n \in \mathbb{N}}$ *is a function sequence on J satisfying* (i) $\{f_n\}_{n \in \mathbb{N}}$ *is bounded on J*; (ii) $\{f_n^{\Delta}\}_{n \in \mathbb{N}}$ *is bounded on J. Then there is a subsequence of* $\{f_n\}_{n \in \mathbb{N}}$ *converges uniformly on J.*

Without loosing generality, we assume that $0, a \in \mathbb{T}$. This paper mainly considers the existence of solutions for the following first-order nonlinear integro-differential system on time scales with periodic boundary value conditions:

$$\begin{cases} x^{\Delta}(t) = f(t, x(Kx)(t)), \; t \in [0, a], \\ x(0) = x(\sigma(a)); \end{cases} \tag{1.1}$$

and "non-periodic" conditions:

$$\begin{cases} x^{\Delta}(t) = f(t, x, (Kx)(t)), \; t \in [0, a]; \\ Ax(0) + Bx(\sigma(a)) = \theta; \end{cases} \tag{1.2}$$

where $(Kx)(t)$ denotes

$$\left(\int_0^t k_1(t, s)x_1(s)\Delta s, \int_0^t k_2(t, s)x_2(s)\Delta s, \cdots, \int_0^t k_n(t, s)x_n(s)\Delta s\right)$$

with $k_i(t, s) : [0, 1] \times [0, 1] \to [0, +\infty)$ continuous for $i = 1, 2, \cdots, n$; A and B

are $n \times n$ matrices with real valued elements, θ is the zero vector in $\mathbb{R}^n$.

For $A = (a_{ij})_{n\times n}$, we denote $\|A\|$ by $(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|)^{\frac{1}{2}}$.

In what follows, we assume function $f : [0, \sigma(a)] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $\det(A + B) \neq 0$.

Noticing that $\det(A + B) \neq 0$, conditions $Ax(0) + Bx(\sigma(a)) = \theta$ do not include the periodic conditions $x(0) = x(\sigma(a))$. Furthermore, if $A = B = I$, where I denotes $n \times n$ identity matrix, then $Ax(0) + Bx(\sigma(a)) = \theta$ reduces to the so called ”anti-periodic” conditions $x(0) = -x(\sigma(a))$. The authors of [4, 6, 8, 20, 22] consider this kind of ”anti-periodic” conditions for differential equations or impulsive differential equations. To the best of our knowledge it is the first paper to deal with integro-differential equations with “anti-periodic” conditions so far.

This paper is organized as follows. Section 1 gives some preliminaries. Section 2 presents some existence theorems for PVPs (1.1) and a couple of examples to illustrate how our newly developed results work. In Section 3 we focus on the existence of solutions for (1.2) and also an example is given.

In what follows, if $x, y \in \mathbb{R}^n$, then $\langle x, y\rangle$ denotes the usual inner product and $\|x\|$ denotes the Euclidean norm of x on $\mathbb{R}^n$. Let

$$C([0,\ \sigma(a)],\ \mathbb{R}^n) = \{x : [0,\ \sigma(a)] \to \mathbb{R}^n,\ x(t) \text{ is continuous}\}$$

with the norm

$$\|x\|_C = \sup_{t\in[0,\ \sigma(a)]} \|x(t)\|.$$

The following well-known fixed-point theorem will be used in the proof of Theorem 3.3.

Lemma 1.4 [19]**.** *Let X be a normed space with $H : X \to X$ a compact mapping. If the set*

$$S := \{u \in X : u = \lambda Hu \ \textit{for some} \ \lambda \in [0, 1)\}$$

is bounded, then H has at least one fixed-point.

2. Existence Results for Periodic Conditions

To begin with, we consider the following periodic boundary value problem

$$\begin{cases} x^{\Delta}(t) + m(t)x(t) = g(t, x(t), (Kx)(t)), \ t \in [0, a]; \\ x(0) = x(\sigma(a)); \end{cases} \tag{2.1}$$

where $g : [0, a] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m : [0, a] \to \mathbb{R}$ are both continuous functions, with m having no zeros in $[0, a]$.

Lemma 2.1. *The BVP* (2.1) *is equivalent to the integral equation*

$$x(t) = e_{-m}(t, 0)\left(\frac{e_{-m}(\sigma(a), 0)}{1 - e_{-m}(\sigma(a), 0)} \int_0^{\sigma(a)} e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau \right.$$

$$\left. + \int_0^t e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau \right), \ t \in J, \tag{2.2}$$

where $\eta(t) = g(t, x(t), (Kx)(t)), \ t \in [0, a]$.

Proof. First note that for any solution of integral equations (2.2), we have by Lemma 1.1 that

$$x^{\Delta}(t) = -m(t)e_{-m}(t, 0)\left(\frac{e_{-m}(\sigma(a), 0)}{1 - e_{-m}(\sigma(a), 0)} \int_0^{\sigma(a)} e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau \right.$$

$$\left. + \int_0^t e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau \right)$$

$$+ e_{-m}(\sigma(t), 0)e_{-m}(0, \sigma(t))\eta(t) = -m(t)x(t) + \eta(t).$$

Moreover, there holds from Lemma 1.2 that

$$x(\sigma(a)) = e_{-m}(\sigma(a), 0)\left(\frac{e_{-m}(\sigma(a), 0)}{1 - e_{-m}(\sigma(a), 0)} \int_0^{\sigma(a)} e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau \right.$$

$$+ \int_0^{\sigma(a)} e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau \Bigg)$$

$$= \frac{e_{-m}(\sigma(a), 0)}{1 - e_{-m}(\sigma(a), 0)} \int_0^{\sigma(a)} e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau = x(0).$$

Now consider equation

$$x^{\Delta}(t) + m(t)x(t) = g(t, x(t), (Kx)(t)). \tag{2.3}$$

Then, by the method of constants variation, we know all solutions of (2.3) can be written as

$$x(t) = e_{-m}(t, 0)\left(\int_0^t e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau + C\right),$$

where $\eta(t) = (t, x(t), (Kx)(t)), t \in [0, a], C \in R^n$ is a constant vector. See that boundary condition $x(0) = x(\sigma(a))$ implies

$$C = \frac{e_{-m}(\sigma(a), 0)}{1 - e_{-m}(\sigma(a), 0)} \int_0^{\sigma(a)} e_{-m}(0, \sigma(\tau))\eta(\tau)\Delta\tau.$$

Thus, the proof is completed.

Lemma 2.2. *The family of* (2.2) *is equivalent to the family of*

$$x(t) = \int_0^{\sigma(a)} G(t, s)g(s, x(s), (Kx)(s))\Delta s,$$

where

$$G(t, s) = \frac{1}{1 - e_{-m}(\sigma(a), 0)} \begin{cases} e_{-m}(t, \sigma(s)), & 0 \le \sigma(s) < t \le \sigma(a); \\ e_{-m}(\sigma(a), 0)e_{-m}(t, \sigma(s)), & 0 \le t \le \sigma(s) \le \sigma(a). \end{cases}$$

Furthermore,

$$|G(t, s)| \le G_0 := \frac{\max\{1, e_{-m}(\sigma(a), 0)\}}{|1 - e_{-m}(\sigma(a), 0)|},$$

$$\forall\, 0 \le s \le a,\ \forall\, 0 \le t \le \sigma(a).$$

Proof. Since the first part is clear, we only prove the second part.

Case 1. $m(t) > 0,\ t \in [0, \sigma(a)]$. We have

$$e_{-m}(\sigma(a), 0) < 1,\ e_{-m}(\sigma(a), 0) \le e_{-m}(t, \sigma(s)) \le 1, \text{ if } 0 \le \sigma(s) < t \le \sigma(a);$$

$$e_{-m}(\sigma(a), 0) \le e_{-m}(\sigma(a), 0)e_{-m}(t, \sigma(s)) = \frac{e_{-m}(\sigma(a), 0)}{e_{-m}(\sigma(s), t)} \le 1 \text{ if } 0 \le \mathrm{t} \le \sigma(s) \le \sigma(a).$$

Thus,

$$0 < \frac{e_{-m}(\sigma(a), 0)}{1 - e_{-m}(\sigma(a), 0)} \le G(t, s) \le \frac{1}{1 - e_{-m}(\sigma(a), 0)}.$$

Case 2. $m(t) < 0,\ t \in [0, \sigma(a)]$. We have

$$e_{-m}(\sigma(a), 0) > 1,\ e_{-m}(\sigma(a), 0) \ge e_{-m}(t, \sigma(s)) \ge 1, \text{ if } 0 \le \sigma(s) < t \le \sigma(a);$$

$$e_{-m}(\sigma(a), 0) \ge e_{-m}(\sigma(a), 0)e_{-m}(t, \sigma(s)) = \frac{e_{-m}(\sigma(a), 0)}{e_{-m}(\sigma(s), t)} \ge 1 \text{ if } 0 \le \mathrm{t} \le \sigma(s) \le \sigma(a).$$

So,

$$G(t, s) < 0 \text{ and } |G(t, s)| \le \frac{e_{-m}(\sigma(a), 0)}{|1 - e_{-m}(\sigma(a), 0)|}.$$

Then the conclusion follows from the combination of case 1 and case 2.

Theorem 2.1. *Let* $g : [0, a] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $m : [0, 1] \to \mathbb{R}$ *be both continuous functions, with m having no zeros in* $[0, a]$. *Assume that there exist constants* $R > 0,\ \alpha \ge 0$ *such that*

$$\frac{\sigma(a)\, G_0 M(R)}{R} < 1 \tag{2.4}$$

and

$$\lambda\|g(t, x, (Kx)(t))\| \le 2\alpha\Big[\langle x, \lambda g(t, x, (Kx)(t))\rangle - m(t)\|x\|^2\Big] + M(R), \tag{2.5}$$

$$\forall \lambda \in [0, 1];\ \forall (t, x) \in [0, a] \times B_R,$$

where $M(R)$ is a positive constant depending on R, $B_R = \{x \in \mathbb{R}^n, \|x\| \leq R\}$. Then PBVP (2.1) has at least one solution $x \in C$ with $\|x\|c < R$.

Proof. Let $C = C([0, \sigma(a)], R^n)$ and $\Omega = \{x(t) \in C, \|x(t)\|c < R\}$. Define an operator $T : \overline{\Omega} \to C$ by

$$Tx(t) = \int_0^{\sigma(a)} G(t, s) g(s, x(s), (Kx)(s)) \Delta s,\ \forall t \in [0, \sigma(a)]. \tag{2.6}$$

Since g, K is continuous and $G(t, s)$ is bounded by G_0, we have that T is continuous and TU is bounded for any bounded set $U \subset C$. Furthermore, for any $\{y_n\}_{n \in \mathbb{N}} \in TU$, there exists a corresponding set $\{x_n\}_{n \in \mathbb{N}} \in U$ such that $y_n = Tx_n$, that is, $y_n^{\Delta}(t) = -m(t) x_n(t)$ $g(t, x_n(t), (Kx_n)(t))$. Taking into account that $\{x_n\}_{n \in \mathbb{N}}$ is uniformly bounded, we have $\{y_n^{\Delta}(t)\}_{n \in \mathbb{N}}$ is also uniformly bounded. Then it follows from Lemma 1.3 and Remark 1.2 that operator T is compact.

Consider that, for the ball Ω,

$$x \neq \lambda Tx,\ \forall x \in C \text{ with } x \in \partial\Omega,\ \forall \lambda \in [0, 1], \tag{2.7}$$

implies

$$0 \notin (I - \lambda T)(x),\ \forall x \in \partial\Omega,\ \forall \lambda \in [0, 1].$$

Define $H_\lambda = I - \lambda T$, $\lambda \in [0, 1]$, where I is the identity. So if (2.5) is true, then from the homotopy principle of Leray-Schauder degree [19, Chap. 4.], we have

$$\begin{aligned} \deg_{LS}(H_{\lambda,}\ \Omega, 0) &= \deg_{LS}(I - \lambda T, \Omega, 0) \\ &= \deg_{LS}(H_{1,}\ \Omega, 0) = \deg_{LS}(H_0, \Omega, 0) \\ &= \deg_{LS}(I, \Omega, 0) = 1 \neq 0. \end{aligned}$$

Therefore, it follows from the non-zero property of Leray-Schauder degree that $H_1(x) = x - Tx = 0$ has at least one $x \in \Omega$.

Now our problem is reduced to prove that (2.7) is true. Observe that the family of problems

$$x = \lambda Tx, \ \lambda \in [0, 1] \tag{2.8}$$

is equivalent to the family of PBVPs

$$\begin{cases} x^{\Delta}(t) + m(t)x(t) = \lambda g(t, x(t), (Kx)(t)), \ t \in [0, a]; \\ x(0) = x(\sigma(a)). \end{cases} \tag{2.9}$$

Consider the function $r(t) = \|x(t)\|^2 t \in [0, \sigma(a)]$, where $x(t)$ is a solution of (2.9). Then $r^{\Delta}(t)$ exists for $t \in [0, a]$ and we have by the product rule

$$\begin{aligned} r^{\Delta}(t) &= \langle x^{\Delta}(t), x(t)\rangle + \langle x(\sigma(t), x^{\Delta}(t))\rangle \\ &= \langle x(t), x^{\Delta}(t)\rangle + \langle x(t) + \mu(t)x^{\Delta}(t), x^{\Delta}(t)\rangle \\ &= 2\langle x(t), x^{\Delta}(t)\rangle + \mu(t)\langle x^{\Delta}(t), x^{\Delta}(t)\rangle \\ &\geq 2\langle x(t), x^{\Delta}(t)\rangle, \ t \in [0, a]. \end{aligned}$$

Let x be a solution of (2.8) with $x \in \overline{\Omega}$. We now show that $x \notin \partial\Omega$. From (2.5) we have, for each $t \in [0, \sigma(a)]$ and each $\lambda \in [0, 1]$,

$$\begin{aligned} \|x(t)\| = \|\lambda Tx(t)\| &= \left\|\int_0^{\sigma(a)} \lambda G(t, s)g(s, x(s), (Kx)(s))\Delta s\right\| \\ &\leq G_0 \int_0^{\sigma(a)} \lambda\|g(s, x(s), (Kx)(s))\|\Delta s \\ &\leq G_0 \int_0^{\sigma(a)} [2\alpha\langle x, \lambda g(s, x(s), (Kx)(s)) - m(s)x(s)\rangle + M(R)]\Delta s \\ &= G_0 \int_0^{\sigma(a)} [2\alpha\langle x(s), x^{\Delta}(s)\rangle + M(R)]\Delta s \end{aligned}$$

$$\leq G_0 \int_0^{\sigma(a)} [\alpha r^{\Delta}(s) + M(R)]\Delta s$$

$$= G_0[\alpha(\| x(\sigma(a))\|^2 - \|x(0)\|^2) + \sigma(a)M(R)]$$

$$= \sigma(a)G_0 M(R).$$

Then it follows from (2.5) that $x \notin \partial\Omega$. Thus, (2.7) is true and the proof is completed.

Corollary 2.1. *Let* g *and* m *be as in Theorem* 2.1 *with* $m(t) < 0$, $t \in [0, a]$. *If there exist constants* $R > 0, \alpha \geq 0$ *such that*

$$\frac{\sigma(a)G_0 M(R)}{R} < 1$$

and

$$\|g(t, x, (Kx)(t))\| \leq 2\alpha[\langle x, g(t, x, (Kx)(t))\rangle - m(t)\|x\|^2] + M(R),$$

$$\forall (t, x) \in [0, a] \times B_R,$$

where $M(R)$ *is a positive constant depending on* R, $B_R = \{x \in \mathbb{R}^n, \|x\| \leq R\}$, *then PBVP* (2.1) *has at least one solution* $x \in C$ *with* $\|x\|_C < R$.

Proof. Multiply both sides of (2.8) by $\lambda \in [0, 1]$ to obtain

$$\lambda\|g(t, x, (Kx)(t))\| \leq 2\alpha[\langle x, \lambda g(t, x, (Kx)(t))\rangle - \lambda m(t)\|x\|^2] + \lambda M(R)$$

$$\leq 2\alpha[\langle x, \lambda g(t, x, (Kx)(t))\rangle - m(t)\|x\|] + M(R),$$

$$\forall (t, x) \in [0, a] \times B_R.$$

It completes the proof.

Now consider the existence of solutions of PBVP (1.1). It is easy to see (1.1) is equivalent to the PBVP

$$x^{\Delta} - m_0 x = f(t, x, (Kx)(t)) - m_0 x, \; t \in [0, a]; \qquad (2.10)$$

$$x(0) = x(\sigma(a)),$$

where $m_0 > 0$ is a constant, $f : [0, a] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Corollary 2.2. *If there exist constants* $R > 0$, $\alpha \geq 0$ *such that*

$$\frac{\sigma(a) e_{m_0}(\sigma(a), 0) M(R)}{e_{m_0}(\sigma(a), 0) - 1} < R \tag{2.12}$$

and

$$\| f(t, x, (Kx)(t)) - m_0 x \| \leq 2\alpha[\langle x, f(t, x, (Kx)(t)) \rangle + m_0 \|x\|^2] + M(R),$$

$$\forall (t, x) \in [0, a] \times B_R,$$

where $M(R)$ *is a positive constant dependent on* R, $B_R = \{x \in \mathbb{R}^n, \|x\| \leq R\}$, *then PBVP* (1.1) *has at least one solution* $x \in C$ *with* $\|x\|_C < R$.

Remark 2.1. If $\mathbb{T} = \mathbb{R}$ and $a = 1$, then (2.12) reduces to

$$\frac{e^{m_0} M(R)}{e^{m_0} - 1} < R.$$

Example 2.1. Consider the following PBVP with $n = 2$, $\mathbb{T} = \mathbb{R}$ and $a = 1$.

$$\begin{cases} x' = 2x + x^3 + \frac{y}{4} \int_0^t (t - s) x(s) ds; \\ y' = 3y + \frac{x}{8} \int_0^1 e^{-ts} y(s) ds + \dfrac{\cos(2\pi t)}{20}; \\ x(0) = x(1), \ y(0) = y(1). \end{cases} \tag{2.13}$$

We prove that (2.13) has at least one solution $(x(t), y(t))^\top$ with $\sqrt{x(t)^2 + y(t)^2} < 0.4$, $\forall t \in [0, 1]$.

First note that (2.13) has no constant solution. Let $u = (x, y)^\top$, $\|u\| = \sqrt{x^2 + y^2}$ and

$$f(t, u, (Ku)(t), (Lu)(t)) = (2x + x^3 + \frac{y}{4} \int_0^t (t - s) x(s) ds, \ 3y$$

$$+ \frac{x}{8}\int_0^t e^{-ts}y(s)ds + \frac{\cos(2\pi t)}{20})^\top.$$

Since $\forall (t, u) \in [0, 1] \times B_R$, $|\frac{y}{4}\int_0^t (t-s)x(s)ds| \leq \frac{R^2}{4}$ and $|\frac{x}{8}\int_0^t e^{-ts} y(s)ds| \leq \frac{R^2}{8}$, we obtain

$$\|F(t, u, (Ku)(t)) - 2u\| = \Big\|(x^3 + \frac{y}{4}\int_0^t (t-s)x(s)ds,\ y$$

$$+ \frac{x}{8}\int_0^t e^{-ts}y(s)ds + \frac{\cos(2\pi t)}{20})^\top)\Big\|$$

$$= \sqrt{\left(x^3 + \frac{y}{4}\int_0^t (t-s)x(s)ds\right)^2 + \left(y + \frac{x}{8}\int_0^t e^{-ts}y(s)ds + \frac{\cos(2\pi t)}{20}\right)^2}$$

$$\leq |x^3 + \frac{y}{4}\int_0^t (t-s)x(s)ds|$$

$$+ |y + \frac{x}{8}\int_0^t e^{-ts}y(s)ds + \frac{\cos(2\pi t)}{20}|$$

$$\leq |x|^3 + \frac{R^2}{4} + |y| + \frac{R^2}{8} + \frac{1}{20},$$

$$= |x|^3 + |y| + \frac{3R^2}{8} + \frac{1}{20} \quad \forall (t, u) \in [0, 1] \times B_R.$$

On the other hand,

$$2\alpha\Big[\langle u,\ F(t,\ u(Ku)(t),\ (Lu)(t))\rangle + 2\|u\|^2\Big]$$

$$= 2\alpha[2x^2 + x^4 + \frac{xy}{4}\int_0^t (t-s)x(s)ds + 3y^2 + \frac{xy}{8}\int_0^t e^{-ts}y(s)ds$$

$$+ \frac{y\cos(2\pi t)}{20} + 2x^2 + 2y^2]$$

$$\geq 2\alpha(4x^2 + x^4 + 5y^2 - \frac{R^3}{4} - \frac{R^3}{8} - \frac{R}{20})$$

$$= 4x^2 + x^4 + 5y^2 - \frac{3R^3}{8} - \frac{R}{20}, \text{ for } \alpha = \tfrac{1}{2}, \ \forall (t, x) \in [0, 1] \times B_R.$$

Clearly,

$$\min_{x \in \mathbb{R}} \{x^4 + 4x^2 - |x|^3\} = 0;$$

$$\min_{x \in \mathbb{R}} \{5y^2 - |y|\} = -\frac{1}{20}.$$

Thus,

$$\|F(t, u, (Ku)(t)) - 2u\| \leq 2\alpha(\langle u, F(t, u, (Ku)(t))\rangle + 2\|u\|^2) + M(R),$$

where

$$\alpha = \frac{1}{2}, \ M(R) = \frac{3R^3}{8} + \frac{3R^2}{8} + \frac{R}{20} + \frac{1}{10}.$$

Now it is sufficient to find a positive constant R satisfying

$$\frac{e^2}{e^2 - 1} M(R) - R < 0.$$

It is not difficult to get $\dfrac{e^2}{e^2 - 1} M(R) - R < 0$ for $R \in [0.4, 0.8]$ by Mathematica 4.0. Then our conclusion follows from Corollary 2.2.

3. Existence Results for "Non-Periodic" Conditions

In this section we study the problem of existence of solutions for BVP (1.2).

Lemma 3.1. *The BVP* (1.2) *is equivalent to the integral equation*

$$x(t) = \int_0^t f(s, x(s), (Kx)(s))\Delta s - (A + B)^{-1} B \int_0^{\sigma(a)} f(s, x(s), (Kx)(s))\Delta s,$$

$$t \in [0, \sigma(a)].$$

Proof. The result can be obtained by direct computation.

Theorem 3.1. *Assume det* $B \neq 0$ *and* $\|B^{-1}A\| \leq 1$. *Suppose there exist constants* $R > 0$, $\alpha \geq 0$ *such that*

$$\sigma(a)(1 + \|(A + B)^{-1} B\|)M(R) < R, \tag{3.1}$$

and

$$\|f(t, x, (Kx)(t))\| \leq 2\alpha\langle x, f(t, x, (Kx)(t))\rangle + M(R), \tag{3.2}$$

$$\forall (t, x) \in [0, a] \times B_R,$$

where $M(R)$ *is a positive constant depending on* $R, B_R = \{x \in \mathbb{R}^n, \|x\| \leq R\}$. *Then BVP* (1.2) *has at least one solution* $x \in C$ *with* $\|x\|_C < R$.

Proof. Let $C = C([0, \sigma(a)], R^n)$ and $\Omega = \{x(t) \in C, \|x(t)\|_C < R\}$. Define an operator $T : \Omega \to C$ by

$$Tx(t) = \int_0^t f(s, x(s), (Kx)(s))ds$$

$$- (A + B)^{-1} B \int_0^{\sigma(a)} f(s, x(s), (Kx)(s))ds, \; t \in [0, \sigma(a)]. \tag{3.3}$$

Since f is continuous, we see that T is also a continuous map. Deducing in a similar way as in the proof of Theorem 2.1, we can still prove that operator T is compact. Then, according to the homotopy principle and non-zero property of Leray-Schauder degree, it is sufficient to prove

$$x \neq \lambda Tx \text{ for all } x \in C \text{ with } \|x\|_C = R \text{ and for all } \lambda \in [0, 1]. \tag{3.4}$$

See that the family of problems

$$x = \lambda Tx, \; \lambda \in [0, 1] \tag{3.5}$$

is equivalent to the family of BVPs

$$\begin{cases} x' = \lambda f(t, x, (Kx)(t)), \; t \in [0, a]; \\ Ax(0) + Bx(\sigma(a)) = \theta. \end{cases} \tag{3.6}$$

Consider function $r(t) = \|x(t)\|^2$, $t \in [0, 1]$, where $x(t)$ is a solution of (3.6). Then $r^{\Delta}(t)$ exists for $t \in [0, a]$ and we have by the product rule

$$\begin{aligned} r^{\Delta}(t) &= \langle x^{\Delta}(t), x(t)\rangle + \langle x(\sigma(t), x^{\Delta}(t))\rangle \\ &= \langle (t), x^{\Delta}(t)\rangle + \langle x(t) + \mu(t)x^{\Delta}(t), x^{\Delta}(t)\rangle \\ &= 2\langle x(t), x^{\Delta}(t)\rangle + \mu(t)\langle x^{\Delta}(t), x^{\Delta}(t)\rangle \\ &\geq 2\langle x(t), x^{\Delta}(t)\rangle, \ t \in [0, a]. \end{aligned}$$

Note that $\|B^{-1}A|\| \leq 1$ implies

$$\|x(\sigma(a))\| = \|B^{-1}Ax(0)\| \leq \|B^{-1}A\| \cdot \|x(0)\| \leq \|x(0)\|.$$

Let x be a solution of (3.5) with. $x \in \overline{\Omega}$. We now show that $x \notin \partial\Omega$. From (3.2) and (3.3) we obtain, for each $t \in [0, \sigma(a)]$ and each $\lambda \in [0, 1]$,

$$\begin{aligned} \|x(t)\| &= \|\lambda Tx(t)\| \\ &= \left\|\int_0^t \lambda f(s, x(s)\ (Kx)(s))\Delta s - (A+B)^{-1}B\int_0^{\sigma(a)} \lambda f(s, x(s)\,(Kx)(s))\Delta s\right\| \\ &\leq (1 + \|(A+B)^{-1}B\|)\int_0^{\sigma(a)} \lambda\|f(s, x(s), (Kx)(s))\|\Delta s \\ &\leq (1 + \|(A+B)^{-1}B\|)\int_0^{\sigma(a)} \|f(s, x(s), (Kx)(s))\|\Delta s \\ &\leq (1 + \|(A+B)^{-1}B\|)\int_0^{\sigma(a)} [2\alpha\langle x, f(s, x(s), (Kx)(s))\rangle + M(R)]\Delta s \\ &\leq (1 + \|(A+B)^{-1}B\|)\int_0^{\sigma(a)} [\alpha(\|x(s)\|^2)^{\Delta} + M(R)]\Delta s \\ &\leq (1 + \|(A+B)^{-1}B\|)\sigma(a)[\alpha(\|x(\sigma(a))\|^2 - \|x(0)\|^2) + M(R)] \\ &\leq \sigma(a)(1 + \|(A+B)^{-1}B\|)M(R). \end{aligned}$$

Then it follows from (3.1) that $x \notin \partial\Omega$. Thus, (3.4) is true and the proof is completed.

Corollary 3.1 *Let f be a scalar-valued function in* (1.2*). and assume there exist constants* $R > 0,\ \alpha \geq 0$ *such that*

$$\frac{3\sigma(a)}{2} M(R) < R, \tag{3.7}$$

a*n*d

$$|f(t,\ x(Kx)(t))| \leq 2\alpha x f(t,\ x,\ (Kx)(t) + M(R),\ \forall(t,\ x) \in [0,\ a] \times B_R, \tag{3.8}$$

where $M(R)$ *is a positive constant depending on* R, $B_R = \{x \in \mathbb{R}^n, |x| \leq R\}$. *Then anti-periodic boundary value problem*

$$\begin{cases} x^{\Delta}(t) = f(t,\ x,\ (Kx)(t)), t \in [0,\ \sigma(a)]; \\ x(0) = -x(\sigma(a)), \end{cases}$$

has at least one solution $x \in C[0,\ \sigma(a)]$ *with* $|x(t)| < R,\ t \in [0,\ \sigma(a)]$.

Proof. Since $A = B = 1$, we have $(A + B)^{-1} = \frac{1}{2}$, $B^{-1}A = 1$, $(1 + \|(A + B)^{-1}B\|) = \frac{3}{2}$. Then the conclusion follows from Lemma 3.1.

Example 3.1. Assume that $\mathbb{T} = \mathbb{R}$ and $a = 1$. Let us show that

$$\begin{cases} x' = x^{\frac{1}{3}} + x^3 + \frac{1}{20}\int_0^t e^{-ts}x(s)ds + \frac{1}{40}\cos(2\pi t), \\ x(0) = -x(1). \end{cases} \tag{3.9}$$

has at least one solution $x(t)$ with $|x(t)| < 1,\ \forall t \in [0,\ 1]$.

Denoting $f(t,\ x,\ (Kx)(t)) = x^{\frac{1}{3}} + x^3 + \frac{1}{0}\int_0^t e^{-ts}x(s)ds + \frac{1}{40}\cos(2\pi t)$,

we see that, for all $(t,\ x) \in [0,\ 1] \times B_R$,

$$|f(t,\ x,\ (Kx)(t))| \leq |x|^{\frac{1}{3}} + |x|^3 + \frac{R}{20} + \frac{1}{40}.$$

On the other hand,

$$2\alpha\langle x,\ f(t,\ x,\ (Kx)(t))\rangle$$

$$= 2\alpha(x^{\frac{4}{3}} + x^4 + \frac{x}{20}\int_0^t e^{-ts}x(s)ds + \frac{x}{40}\cos(2\pi t)$$

$$= 2x^{\frac{4}{3}} + 2x^4 + \frac{x}{10}\int_0^t e^{-ts}x(s)ds + \frac{x}{20}\cos(2\pi t),\ \text{for}\ \alpha = 1$$

$$\geq 2x^{\frac{4}{3}} + 2x^4 - \frac{R^2}{10} - \frac{R}{20}.$$

Since

$$\min_{x\in\mathbb{R}}\{x^{\frac{4}{3}} + 2x^4 - |x|^{\frac{1}{3}} - |x|^3\} \geq -0.4,$$

we choose

$$M(R) = \frac{R^2}{10} + \frac{R}{10} + 0.425.$$

Then

$$\|f(t,\ x,\ (Kx)(t))\| \leq 2\langle x,\ f(t,\ s,\ (Kx)(t))\rangle + M(R).$$

It is not difficult to check that $\frac{3}{2}M(R) < R$ for $R \in [1,\ 4]$. So, the conclusion follows from Corollary 3.1.

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